

Ch. Pair of straight lines:

Thm. A homogeneous equation of the second degree, namely

$$ax^2 + 2hxy + by^2 = 0 \quad \text{--- (1)}$$

represents a pair of straight lines through the origin.

Pf-

Dividing (1) by  $x^2$ ,

$$a + 2h \cdot \frac{y}{x} + b \left(\frac{y}{x}\right)^2 = 0$$

$$\therefore b \left(\frac{y}{x}\right)^2 + 2h \cdot \left(\frac{y}{x}\right) + a = 0. \quad \text{--- (2)}$$

This is a quadratic eqn in  $\left(\frac{y}{x}\right)$ . Hence it will have two roots. Let the roots be  $m_1$  and  $m_2$ .

$$\therefore m_1 + m_2 = -\frac{2h}{b}, \quad m_1 m_2 = \frac{a}{b}$$

l.h.s. of (2) becomes

 --- (3)

Now (2) can be written as

$$b \left(\frac{y}{x}\right)^2 + 2h \left(\frac{y}{x}\right) + a = b \left(\frac{y}{x} - m_1\right) \left(\frac{y}{x} - m_2\right).$$

$\therefore$  (2) becomes,

$$b \left(\frac{y}{x} - m_1\right) \left(\frac{y}{x} - m_2\right) = 0$$

$$\Rightarrow (y - m_1 x)(y - m_2 x) = 0$$

$\therefore$  either  $y - m_1 x = 0$  or  $y - m_2 x = 0$

$$\therefore y = m_1 x \quad \text{or} \quad y = m_2 x. \quad \text{--- (4)}$$

(4) shows that the eqn (1) represents

a pair of straight lines passing through the origin whose individual equations are  $y = m_1 x$  &  $y = m_2 x$ ,

$$\text{where } m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

E Angle between the pair of lines given by the equation

$$ax^2 + 2hxy + by^2 = 0.$$

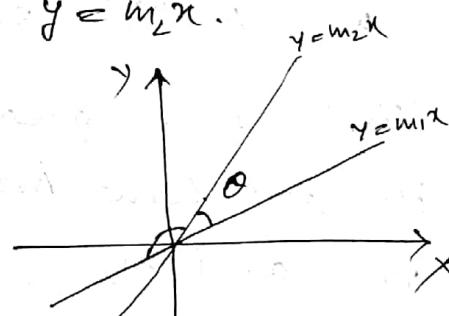
Soln. Let the individual equations of the two lines be

$$y = m_1 x \text{ and } y = m_2 x.$$

Let the

$$\therefore m_1 + m_2 = -\frac{2h}{b},$$

$$\text{and } m_1 m_2 = \frac{a}{b}.$$



Let  $\theta$  be the angle between the two lines.

$$\begin{aligned} \tan \theta &= \frac{m_1 - m_2}{1 + m_1 m_2} \\ &= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} \\ &= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4 \cdot \frac{a}{b}}}{1 + \frac{a}{b}} \\ &= \frac{\sqrt{\frac{4(h^2 - ab)}{b^2}}}{\frac{b+a}{b}} \\ &= \frac{\frac{2\sqrt{h^2 - ab}}{b}}{\frac{a+b}{b}} \end{aligned}$$

$$\Rightarrow \tan \alpha = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\Rightarrow \alpha = \tan^{-1} \left( \frac{2\sqrt{h^2 - ab}}{a+b} \right).$$

Deduction: If the two lines be perpendicular to each other, then  $\alpha = \pi/2$ .

$$\therefore \tan \pi/2 = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\Rightarrow \alpha = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\Rightarrow a+b=0.$$

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14/11/06

To find the equation of the bisectors of the angles between the pair of lines given by

$$ax^2 + 2hxy + by^2 = 0.$$

$\frac{\partial}{\partial x}$

The given eqn is

$$ax^2 + 2hxy + by^2 = 0 \quad \text{--- (1)}$$

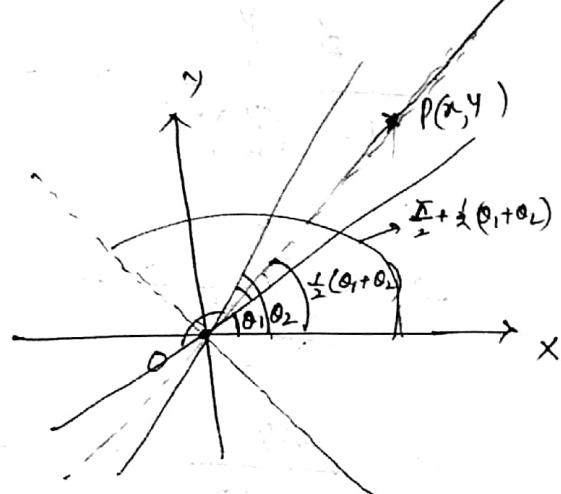
Let the individual eqns of the two lines be

$$y = m_1 x \text{ and } y = m_2 x.$$

$$\begin{aligned} \text{Then } m_1 + m_2 &= -\frac{2h}{b}, \\ m_1 m_2 &= \frac{a}{b} \end{aligned} \quad \text{--- (2)}$$

Let  $\alpha_1$  and  $\alpha_2$  be the angles which the two lines make with the x-axis.

$$\therefore \tan \alpha_1 = m_1 \text{ and } \tan \alpha_2 = m_2, \text{ say.}$$



If  $\theta$  be the angle which either bisector (internal or external) makes with the x-axis, then

$$\theta = \frac{1}{2}(\theta_1 + \theta_2) \quad (\text{for internal})$$

$$\text{or } \theta = \frac{\pi}{2} + \frac{1}{2}(\theta_1 + \theta_2) \quad (\text{for external})$$

$$\left. \begin{aligned} \theta &= \theta_1 + \frac{1}{2}(\theta_2 - \theta_1) \\ &= \frac{1}{2}(2\theta_1 + \theta_2 - \theta_1) \\ &= \frac{1}{2}(\theta_1 + \theta_2) \quad (\text{for internal}) \end{aligned} \right\}$$

$\therefore$  In either case,

$$\tan 2\theta = \tan(\theta_1 + \theta_2)$$

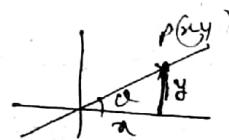
$$\Rightarrow \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 + m_2}{1 - m_1 m_2}$$

$$\Rightarrow \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{-\frac{2h}{b}}{1 - \frac{a}{b}} \rightarrow (3) \quad [\text{by } (2)]$$

If  $(x, y)$  be any point on the bisector, then

$$\tan\theta = \frac{y}{x}$$



$\therefore$  Putting  $\tan\theta = \frac{y}{x}$  in (3) we get,

$$\frac{2/\frac{y}{x}}{1 - (\frac{y}{x})^2} = \frac{-\frac{2h}{b}}{1 - \frac{a}{b}}$$

$$\Rightarrow \frac{\frac{2x}{y}}{1 - \frac{y^2}{x^2}} = \frac{-\frac{2h}{b}}{\frac{b-a}{b}}$$

$$\Rightarrow \frac{xy}{x^2 - y^2} = -\frac{h}{a-b}$$

$$\Rightarrow \frac{xy}{\frac{a-b}{h}} = \frac{x^2 - y^2}{a-b}$$

$$\Rightarrow \frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

which is the required equation of the bisector.

To find the condition that the pair of linear general eqn of second degree in  $x$  and  $y$  represents a pair of st. lines.

Soln Let the general equation of second degree be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

If (1) represents a pair of st. lines, then l.h.s. of (1) can be factorised into two linear factors.

Let us express eqn (1) as a quadratic eqn in  $x$  as

$$ax^2 + 2(ky+g)x + (by^2 + 2fy + c) = 0, a \neq 0.$$

$$\Rightarrow x = \frac{-2(ky+g) \pm \sqrt{4(ky+g)^2 - 4 \cdot a \cdot (by^2 + 2fy + c)}}{2a}$$

If l.h.s. of (1) can be expressed as the product of two linear factors, then the roots must be real and rational.

For this, we must have

$$4(ky+g)^2 - 4a(by^2 + 2fy + c) \geq 0$$

and a perfect square

$$(ky+g)^2 - a(by^2 + 2fy + c)$$

$$= (k^2 - ab)y^2 + 2(kg - af)y + g^2 - ac$$

must be a perfect square.

for this, we must have,

$$(k^2 - ab)(g^2 - ac) = (kg - af)^2$$

$$\Rightarrow (k^2 - ab)(g^2 - ac) - (kg - af)^2 = 0$$

$$\Rightarrow k^2g^2 - k^2ac - abg^2 + a^2bc - kg^2 + 2agh - af^2 = 0.$$

$$\Rightarrow a(abc + 2gh - af^2 - bg^2 - ch^2) = 0.$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad [ \because a \neq 0 ]$$

————— (A)

i.e.,

$$\begin{vmatrix} a & gh & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

To find the condition that the general eqn of second degree represent a pair of intersecting st. lines:

Let the general eqn of second degree be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots \textcircled{1}$$

and it represents a pair of st. lines intersecting at  $(x', y')$ .

Let us transfer the origin to the pt  $(x', y')$  without changing the direction of the axes.

The transformed eqn is given by,

$$a(x+x')^2 + 2h(x+x')(y+y') + b(y+y')^2 + 2g(x+x') + 2f(y+y') + c = 0. \quad \textcircled{2}$$

$\left\{ \begin{array}{l} x = x' + h \\ y = y' + k \end{array} \right.$

This eqn now represents a pair of st. lines passing through the new origin  $(x', y')$ . Hence it must be homogeneous in  $x$  and

y.

$\therefore$  coefficients of  $x$  and  $y$  and the constant term must vanish.

$$\text{coeff. of } x = 2(ax' + by' + f) = 0$$

$$\therefore " y = 2(hx' + by' + f) = 0$$

$$\text{and constant term} = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0$$

$$ax' + by' + f = 0 \quad \text{--- (3)}$$

$$hx' + by' + f = 0 \quad \text{--- (4)}$$

$$\text{and } x'(ax' + by' + f) + y'(hx' + by' + f) + (gx' + fy' + c) = 0$$

$$\Rightarrow x' \cdot 0 + y' \cdot 0 + gx' + fy' + c = 0 \quad (\text{by (3) \& (4)})$$

$$\Rightarrow gx' + fy' + c = 0 \quad \text{--- (5)}$$

Eliminating  $x'$ ,  $y'$  from (3), (4) and (5)

we get,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

which is same as (A).

Deduction: From (3) & (4), ~~after we get~~

$$\frac{x'}{h - bg} = \frac{-y'}{at - gh} = \frac{1}{ab - h^2}$$

$$\Rightarrow x' = \frac{ht - bg}{ab - h^2}, \quad y' = \frac{gh - at}{ab - h^2}$$

∴ The point of intersection is

$$\left( \frac{af-bg}{ab-h^2}, \frac{gh-af}{ab-h^2} \right).$$

16/11/06

2 To find the distance of the point of intersection from the origin:

We know that the point of intersection of the two lines represented by  $ax^2+2hxy+by^2+2gx+2fy+c=0$ .

$$i) \left( \frac{af-bg}{ab-h^2}, \frac{gh-af}{ab-h^2} \right).$$

Let  $d$  be the distance from the origin.

$$\begin{aligned} \text{Then } d^2 &= \left( \frac{af-bg}{ab-h^2} - 0 \right)^2 + \left( \frac{gh-af}{ab-h^2} - 0 \right)^2 \\ &= \frac{(af-bg)^2 + (gh-af)^2}{(ab-h^2)^2} \\ &= \frac{a^2f^2 - 2afbg + b^2g^2 + g^2h^2 - 2afgh + a^2f^2}{(ab-h^2)^2} \\ &= \frac{a^2f^2 + b^2g^2 + g^2h^2 + a^2f^2 - (a+b)2afgh}{(ab-h^2)^2} \end{aligned}$$

$$\begin{aligned} \therefore abf^2 + 2afgh - af^2 - bg^2 - ch^2 &= 0 \\ \Rightarrow 2afgh &= af^2 + bg^2 + ch^2 - abc. \end{aligned}$$

$$\begin{aligned} &= \frac{a^2f^2 + b^2g^2 + g^2h^2 + a^2f^2 - (a+b)(af^2 + bg^2 + ch^2 - abc)}{(ab-h^2)^2} \\ &= \frac{a^2f^2 + b^2g^2 + g^2h^2 + a^2f^2 - af^2 - bg^2 - ch^2 - abc - abf^2 - b^2g^2 - bch^2 + abc}{(ab-h^2)^2} \\ &= \frac{-f^2(ab-h^2) - g^2(ab-h^2) + ac(ab-h^2) + bc(ab-h^2)}{(ab-h^2)^2} \\ &= \frac{(ab-h^2)\{c(a+b) - f^2 - g^2\}}{(ab-h^2)^2} \end{aligned}$$

$$\Rightarrow d^2 = \frac{c(a+b) - f^2 - g^2}{ab - h^2}$$

$$\Rightarrow d = \sqrt{\frac{c(a+b) - f^2 - g^2}{ab - h^2}}$$

To find the angle between the lines represented by the general equation of second degree i.e.,  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ . ————— (1)

~~so~~<sup>n</sup> Let the individual equations of the two lines given by (1) be

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2$$

Then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = b(y - m_1 x - c_1)(y - m_2 x - c_2)$$

∴ Equating the coefficients of the like powers of  $x$  and  $y$ , we get,

$$m_1 + m_2 = -\frac{2f}{b}, \quad m_1 m_2 = \frac{a}{b}, \quad m_1 c_2 + m_2 c_1 = \frac{2g}{b},$$

$$c_1 + c_2 = -\frac{2h}{b}, \quad c_1 c_2 = \frac{c}{b}.$$

If  $\theta$  be the angle between the lines, then

$$\begin{aligned} \tan \theta &= \frac{m_1 - m_2}{1 + m_1 m_2} \\ &= \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2} \end{aligned}$$

$$= \frac{\sqrt{\left(\frac{2f}{b}\right)^2 - 4 \cdot \frac{a}{b}}}{1 + \frac{a}{b}}$$

$$\Rightarrow \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\therefore \theta = \tan^{-1} \left( \frac{2\sqrt{h^2 - ab}}{ab} \right).$$

Deduction: If the two lines be perpendicular to each other, then  $\theta = 90^\circ$ .

$$\Rightarrow \tan \theta = \tan 90^\circ$$

$$\Rightarrow \frac{2\sqrt{h^2 - ab}}{ab} = \infty$$

$$\Rightarrow a + b = 0.$$

$$\text{i. coeff. of } x^2 + \text{coeff. of } y^2 = 0$$

Pair of parallel st. lines:

$$\text{Let } ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

represents a pair of parallel st. lines. Also let the individual equations of the lines be

$$y = mx + c_1 \quad \text{and} \quad y = mx + c_2.$$

Then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = b(y - mx - c_1)(y - mx - c_2)$$

$$2m = -\frac{2h}{b}, \quad m^2 = \frac{a}{b}, \quad m(c_1 + c_2) = \frac{2g}{b}, \quad \text{--- (4)}$$

$$c_1 + c_2 = -\frac{2f}{b}, \quad c_1 c_2 = \frac{c}{b}. \quad \text{--- (5)}$$

$$\text{Now, } \text{--- (2) \& (3) } \Rightarrow \frac{a}{b} = \left(-\frac{h}{b}\right)^2$$

$$\Rightarrow \frac{a}{b} = \frac{h^2}{b^2}$$

$$\Rightarrow h^2 = ab$$

$$\text{or } \frac{a}{h^2} = \frac{b}{ab} \quad \text{--- (6)}$$

From (4) & (5),

$$\text{then } \frac{m(c_1 + c_2)}{(a + c_2)} = \frac{\frac{2y}{b}}{-\frac{2f}{b}}$$

$$\Rightarrow m = -\frac{g}{f}$$

$$\Rightarrow -\frac{h}{b} = -\frac{g}{f} \quad (\text{from } ②)$$

$$\Rightarrow \frac{h}{b} = \frac{g}{f} . \quad ③$$

$$\text{or } hf = bg$$

$$\Rightarrow hf^2 = bg^2$$

$$\Rightarrow abf^2 = bg^2 \quad (\text{from } ⑦)$$

$$\Rightarrow af^2 = bg^2.$$

$\therefore$  From ⑦ & ⑧ we get that-

$$\frac{a}{b} = \frac{h}{b} = \frac{g}{f}$$

which is the required condition.

Note: Also  $h = ab$ ,  $hf = bg$  and  $af^2 = bg^2$  may be taken as the condition of parallelism.